J Glob Optim (2010) 46:589–601
DOI 10.1007/s10898-009-9444-9

# Reduction of finite exhausters

**Jerzy Grzybowski · Diethard Pallaschke · Ryszard Urbański**

Received: 10 May 2009 / Accepted: 11 May 2009 / Published online: 26 May 2009


**Abstract** In this paper we introduce the notation of shadowing sets which is a generalization of the notion of separating sets to the family of more than two sets. We prove that $\bigcap_{i\in I} A_i$ is a shadowing set of the family $\{A_i\}_{i\in I}$ if and only if $\sum_{i\in I} A_i = \bigvee_{i\in I} \sum_{k\in I\setminus\{i\}} A_i + \bigcap_{i\in I} A_i$. It generalizes the theorem stating that $A \cap B$ is separating set for $A$ and $B$ if and only if $A + B = A \cap B + A \vee B$. In terms of shadowing sets, we give a criterion for an arbitrary upper exhauster to be an exhauster of sublinear function and a criterion for the minimality of finite upper exhausters. Finally we give an example of two different minimal upper exhausters of the same function, which answers a question posed by Vera Roshchina (J Convex Anal, to appear).

**Keywords** Minkowski–Rådström–Hörmander spaces · Exhausters · Pairs of closed bounded convex sets

**Mathematics Subject Classification (2000)** 46B20 · 52A05 · 54B20

## 1 Introduction

The main object of this paper is to study the theory of exhausters, which was introduced by V.F. Demyanov and A.M. Rubinov [12] in quasi-differential calculus and subsequently

Dedicated to the 75th Birthday of Professor Franco Gianessi.

J. Grzybowski · R. Urbański
Faculty of Mathematics and Computer Science, Adam Mickiewicz University,
Umultowska 87, 61-614 Poznań, Poland
e-mail: jgrz@amu.edu.pl

R. Urbański
e-mail: rich@amu.edu.pl

D. Pallaschke (✉)
Institute of Operations Research, University of Karlsruhe, Kaiserstr.12, 76128 Karlsruhe, Germany
e-mail: lh09@rz.uni-karlsruhe.de

studied in a series of papers by V.F. Demyanov, A.M. Rubinov and V. Roshchina (see [5–11,27–29]) in the framework of lattices and semigroups (see [1,3,20]). We begin with the following statement:

**Proposition 1.1** *([20,31])Let $S = (S, +)$ be a commutative semigroup with zero, satisfying the cancellation law*

$$a + s = b + s \text{ implies } a = b \tag{cl}$$

*We define the relation of equivalence $\sim$ in $S^2 = S \times S$ by*

$$(a, b) \sim (c, d) \text{ if and only if } a + d = b + c.$$

*Let us denote*

$$[a, b] = \{(c, d) \in S^2 | (c, d) \sim (a, b)\},$$

*$(a, b) \in S^2$ and $\widetilde{S} = S^2/_\sim$. Then $(\widetilde{S}, +)$ is a commutative group with*

$$[a, b] + [c, d] = [a + c, b + d],\ \widetilde{0} = [a, a], -[a, b] = [b, a]$$

*and $j(a) = [a, 0]$ defines an isomorphic mapping embedding $S$ in $\widetilde{S}$; $\widetilde{S} = j(S) - j(S)$. Moreover, if G is a commutative group such that S is embedded in G in an isomorphic way then $\widetilde{S}$ is isomorphic to some subgroup of G.*

Let $(S, +, \leq)$ be a commutative ordered semigroup with zero. For $a, b \in S$ let $a \vee b = \sup\{a, b\}$, $a \wedge b = \inf\{a, b\}$, if they exist. Assume that $S$ satisfies the order cancellation law.

$$a + s \leq b + s \text{ implies } a \leq b \tag{olc}$$

and the addition is isotonic

$$a \leq b \text{ implies } a + s \leq b + s. \tag{i}$$

Also assume that

$$\text{for any } a, b \in S \text{ there exists } a \vee b. \tag{v}$$

Let distributive law be satisfied:

$$a \vee b + s = (a + s) \vee (b + s). \tag{d}$$

Obviously, the order cancellation law implies the cancellation law. Let us introduce an ordering in $\widetilde{S}$ in the following way:

$$[a, b] \leq [c, d] \text{ if and only if } a + d \leq b + c.$$

It is easy observe that the definition does not depend on the choice of representatives.

**Proposition 1.2** *If $S = (S, +, \leq)$ is an ordered commutative semigroup with zero, isotonic, satisfying the order cancellation law, distributive law and having the* sup *property (v) then $\widetilde{S} = (\widetilde{S}, +, \leq)$ is an ordered group such that for $\tilde{x} = [a, b]$, $\tilde{y} = [c, d]$ we have*

$$\tilde{x} \vee \tilde{y} = \sup\{\tilde{x}, \tilde{y}\} = [(a + d) \vee (b + c), b + d],$$
$$\tilde{x} \wedge \tilde{y} = \inf\{\tilde{x}, \tilde{y}\} = [a + b, (a + d) \vee (b + c)].$$

*Proof* Let $\tilde{x} = [a, b]$, $\tilde{y} = [c, d]$. Notice that

$$a + b + d = b + a + d \leq b + (a + d) \vee (b + c).$$

Hence $\tilde{x} \leq [(a + d) \vee (b + c), b + d] = \tilde{z}$. Analogously, $\tilde{y} \leq \tilde{z}$. Let $\tilde{x}, \tilde{y} \leq [e, f]$. Then $a + f \leq b + e$ and $c + f \leq d + e$. Hence $a + d + f \leq b + d + e$ and $b + c + f \leq b + d + e$. Therefore, $(a + d + f) \vee (b + c + f) \leq b + d + e$. But

$$(a + d + f) \vee (b + c + f) = (a + d) \vee (b + c) + f$$

which implies $\tilde{z} \leq [e, f]$. Therefore, $\tilde{x} \vee \tilde{y} = \tilde{z}$.

Notice that

$$\begin{aligned} \tilde{x} \wedge \tilde{y} &= -\sup\{-\tilde{x}, -\tilde{y}\} = -([b, a] \vee [d, c]) \\ &= -[(a + d) \vee (b + c), a + c] = [a + c, (a + d) \vee (b + c)]. \end{aligned}$$ □

Let $X$ be a Hausdorff topological vector space over the field $\mathbb{R}$ and $\mathcal{B}(X)$ the family of all nonempty closed bounded convex subsets of $X$.

*Example 1.3* Let $X = \mathbb{R}^2$ and $S = \mathcal{B}_0(X)$, where $\mathcal{B}_0(X) \subset \mathcal{B}(X)$ is the family of nonempty compact convex subsets of the plane containing zero. If we define the operation "+" by the Minkowski addition and the partial order "$\leq$" by $A \leq B$ if and only if $B \subset A$, then $(\mathcal{B}_0(X), +, \leq)$ becomes a partially ordered commutative semigroup which satisfies order cancellation law and isotony condition. Moreover for $A, B \in \mathcal{B}_0(X)$, we have $A \vee B = A \cap B$. Now we observe that $A \cap B + C \subset (A + C) \cap (B + C)$. Now we consider $A = (0, -1) \vee (0, 1)$, $B = (-1, 0) \vee (1, 0)$ and the unit ball $C = \mathbb{B}((0, 0), 1)$. Then $A \cap B = \{(0, 0)\}$ and $A \cap B + C = \mathbb{B}((0, 0), 1)$. But $(A + C) \cap (B + C) = A + B = (-1, 1) \vee (-1, -1) \vee (1, -1) \vee (1, 1)$. Hence the semigroup $S$ satisfies the conditions (olc),(i),(v) but not satisfies the distribution law (d) (Fig. 1).

*Remark 1.4* In $\mathbb{R}^2$ for arbitrary $A, B, C \in \mathcal{B}(\mathbb{R}^2)$, $A \cap B \neq \emptyset$ the intersection $A \cap B$ is a summand of $(A + C) \cap (B + C)$(see [18]).

It is easy to observe that the following Lemma is true:

**Lemma 1.5** *Let $G = (G, +, \leq)$ be a commutative order group and $(x_i)_{i \in I} \subset G$. If $\sup_{i \in I} x_i \in G$, then for every $x \in G$,*

$$\sup_{i \in I}(x_i + x) = \sup_{i \in I} x_i + x.$$

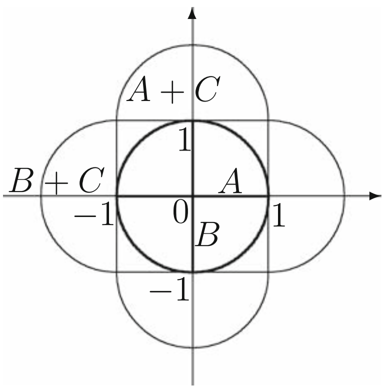



**Fig. 1** Invalidity of the distribution law

A commutative partially ordered semigroup $S$ with zero which satisfies the conditions (olc), (i), (v) and (d) is called *q-semigroup*. For a general representation theorem of q-semigroups we refer also to the paper of H. Ratschek and G. Schröder [26].

**Lemma 1.6** *Let $S = (S, +, \leq)$ be a commutative ordered q-semigroup. Then for $\tilde{x}_i = [a_i, 0]$, $i \in I$, I-finite*

$$\sup_{i\in I} \tilde{x}_i = \left[\sup_{i\in I} a_i, 0\right].$$

*Proof* Let $a = \sup_{i\in I} a_i$, $\tilde{x}_i = [a_i, 0]$. Since $a_i \leq a$ for $i \in I$, $\tilde{x}_i \leq \tilde{x}$. Now let $\tilde{x}_i \leq \tilde{y} = [b, c]$, then $a_i + c \leq b$ for $i \in I$ and we have $\sup_{i\in I}(a_i + c) = a + c \leq b$, which imply $\tilde{x} \leq \tilde{y}$. Hence $\tilde{x} = \sup_{i\in I} \tilde{x}_i$. □

**Theorem 1.7** *Let $S = (S, +, \leq)$ be a commutative ordered q-semigroup and $([a_i, b_i])_{i\in I} \subset \tilde{S}$, I-finite. Then*

$$\sup_{i\in I}[a_i, b_i] = \left[\bigvee_{i\in I}\left(a_i + \sum_{k\in I\setminus\{i\}} b_k\right), \sum_{i\in I} b_i\right],$$

$$\inf_{i\in I}[a_i, b_i] = \left[\sum_{i\in I} a_i, \bigvee_{i\in I}\left(b_i + \sum_{k\in I\setminus\{i\}} a_k\right)\right].$$

*Proof* Let $\tilde{x}_i = [a_i, b_i]$ and denote $\tilde{x} = [\sum_{i\in I} b_i, 0]$, $i \in I$. Then by Lemma 1.5 and Lemma 1.6, we obtain

$$\begin{aligned}
\sup_{i\in I} \tilde{x}_i &= \sup_{i\in I}(\tilde{x}_i + \tilde{x} - \tilde{x}) = \sup_{i\in I}(\tilde{x}_i + \tilde{x}) - \tilde{x} \\
&= \sup_{i\in I}\left([a_i, 0] - [b_i, 0] + \left[\sum_{i\in I} b_i, 0\right]\right) - \tilde{x} \\
&= \sup_{i\in I}\left[a_i + \sum_{k\in I\setminus\{i\}} b_k, 0\right] - \tilde{x} \\
&= \left[\bigvee_{i\in I}\left(a_i + \sum_{k\in I\setminus\{i\}} b_k\right), \sum_{i\in I} b_i\right].
\end{aligned}$$

From the equations $\inf_{i\in I} \tilde{x}_i = -\sup_{i\in I} -\tilde{x}_i$, we obtain the second formulas. □

**Theorem 1.8** *Let $S = (S, +, \leq)$ be a commutative ordered q-semigroup and $([a_i, b_i])_{i\in I} \subset \tilde{S}$, $J \subset I$-finite. Then $\inf_{i\in J}[a_i, b_i] = \inf_{i\in I}[a_i, b_i]$ if and only if*

$$\bigvee_{i\in J}\left(b_i + \sum_{k\in I\setminus\{i\}} a_k\right) = \bigvee_{i\in I}\left(b_i + \sum_{k\in I\setminus\{i\}} a_k\right).$$

*Proof* Let $\tilde{x}_i = [a_i, b_i]$, $i \in I$. Then $\inf_{i\in J} \tilde{x}_i = \inf_{i\in I} \tilde{x}_i$ if and only if

$$\left[\sum_{i\in J} a_i, \bigvee_{i\in J}\left(b_i + \sum_{k\in J\setminus\{i\}} a_k\right)\right] = \left[\sum_{i\in I} a_i, \bigvee_{i\in I}\left(b_i + \sum_{k\in I\setminus\{i\}} a_k\right)\right],$$

it is equivalent

$$\sum_{i\in I\setminus J} a_i + \bigvee_{i\in J}\left(b_i + \sum_{k\in J\setminus\{i\}} a_k\right) = \bigvee_{i\in J}\left(b_i + \sum_{k\in I\setminus\{i\}} a_k\right)$$
$$= \bigvee_{i\in I}\left(b_i + \sum_{k\in I\setminus\{i\}} a_k\right). \qquad \square$$

## 2 Shadowing

For a semigroup S we say that an element $c \in S$ *shadows* $(\tilde{x}_i)_{i\in I} \subset \tilde{S}$, if $\inf_{i\in I} \tilde{x}_i \le [c, 0]$. If $\inf_{i\in I} \tilde{x}_i = [c, 0]$ then we say that element $c$ *exactly shadows* $(\tilde{x}_i)_{i\in I}$. In the case of a $q$-semigroup an abstract difference has been recently introduced [22]. Let $a, b \in S$ be two elements of a $q$-semigroup. Then *abstract difference* of $a$ and $b$ is the greatest element (if it exists) of the set $\mathcal{D}(a, b) = \{x \mid x + b \le a\}$ and denoted by $a \dot{-} b$.

The abstract difference [22] has the following properties:

(D1) If $a \dot{-} b$ exists, then $(a \dot{-} b) + b \le a$.
(D2) If $a = b + c$, then $c = a \dot{-} b$.
(D3) For every $a \in S$, we have $a \dot{-} a = 0$.
(D4) If $a \le b$ and for some $c \in S$, $a \dot{-} c$, $b \dot{-} c$ exist, then $a \dot{-} c \le b \dot{-} c$.
(D5) If $a \dot{-} b$ exists then for every $c \in S$, $(a + c) \dot{-} (b + c) = a \dot{-} b$.

For $(\tilde{x}_i)_{i\in I} \subset \tilde{S}$, $\tilde{x}_i = [a_i, b_i]$,we denote

$$\bigvee_{i\in I}^{|I|-1} \tilde{x}_i = \bigvee_{i\in I}\left(b_i + \sum_{k\in I\setminus\{i\}} a_k\right),$$

in the case when $\tilde{x}_i = [a_i, 0]$ we write

$$\bigvee_{i\in I}^{|I|-1} \tilde{x}_i = \bigvee_{i\in I}^{|I|-1} a_i = \bigvee_{i\in I}\left(\sum_{k\in I\setminus\{i\}} a_k\right).$$

**Theorem 2.1** *If S is a commutative ordered q-semigroup, then element $c \in S$ exactly shadows $(\tilde{x}_i)_{i\in I} \subset \tilde{S}$,if and only if $\bigvee_{i\in I}^{|I|-1} \tilde{x}_i$ is a summand of $\sum_{i\in I} a_i$.*

*Proof* Let $\tilde{x}_i = [a_i, b_i]$, $i \in I$. Then $\inf_{i\in I} \tilde{x}_i = [\sum_{i\in I} a_i, \bigvee_{i\in I}^{|I|-1} \tilde{x}_i] \in j(S)$ if and only if $\sum_{i\in I} a_i = \bigvee_{i\in I}^{|I|-1} \tilde{x}_i + c$ for some $c \in S$. $\square$

Now we introduce the following condition:

$$\text{If } \mathcal{D}(a, b) \ne \emptyset, \text{ then } a \dot{-} b \text{ exists.} \qquad \text{(s)}$$

A commutative partially ordered $q$-semigroup $S$ which satisfies condition (s), is called a *$\dot{q}$-semigroup*.

**Lemma 2.2** *If S is a commutative ordered$\dot{q}$-semigroup and $(\tilde{x}_i)_{i\in I} \subset \tilde{S}$. Then $\bigvee_{i\in I}^{|I|-1} \tilde{x}_i$ is a summand of $\sum_{i\in I} a_i$ if and only if $\bigwedge_{i\in I}(a_i \dot{-} b_i)$ exactly shadows $(\tilde{x}_i)_{i\in I}$.*

*Proof* Let for a some $c \in S$, $c$ exactly shadows$\tilde{x}_i = [a_i, b_i]$, $i \in I$. Then

$$\sum_{i\in I} a_i = \bigvee_{i\in I}^{|I|-1} \tilde{x}_i + c \geq b_i + \sum_{i\in I\setminus\{i\}} a_i + c, \quad i \in I.$$

Therefore $a_i \geq b_i + c$ and from condition (s) it follows that the difference $a_i \dot{-} b_i$ exists and moreover $a_i \dot{-} b_i \geq c$. Hence $c \leq \bigwedge_{i\in I}(a_i \dot{-} b_i)$. Now we observe that for every $k \in I \setminus \{i\}$ we have

$$b_i + \sum_{k\in I\setminus\{i\}} a_k + \bigwedge_{i\in I}(a_i \dot{-} b_i) \leq \sum_{k\in I\setminus\{i\}} a_k + b_i + (a_i \dot{-} b_i) \leq \sum_{i\in I} a_i.$$

Therefore

$$\bigvee_{i\in I}^{|I|-1} \tilde{x}_i + \bigwedge_{i\in I}(a_i \dot{-} b_i) \leq \bigvee_{i\in I}^{|I|-1} \tilde{x}_i + c,$$

and from the order cancellation law $\bigwedge_{i\in I}(a_i \dot{-} b_i) \leq c$. Hence $c = \bigwedge_{i\in I}(a_i + c \dot{-} b_i)$. □

**Theorem 2.3** *If S is a commutative ordered* $\dot{q}$*-semigroup and* $(c, d) \in [\bigvee_{i\in I}^{|I|-1} \tilde{x}_i, \sum_{i\in I} a_i]$, *then* $\bigwedge_{i\in I}(a_i + c \dot{-} b_i) = d$ *exactly shadows* $([a_i + c, b_i])_{i\in I}$.

*Proof* Let $c + \sum_{i\in I} a_i = \bigvee_{i\in I}^{|I|-1} \tilde{x}_i + d$. We observe that

$$\bigvee_{i\in I}\left(b_i + \sum_{k\in I\setminus\{i\}} (a_k + c)\right) = \bigvee_{i\in I}\left(b_i + (|I| - 1)c + \sum_{k\in I\setminus\{i\}} a_k\right)$$
$$= (|I| - 1)c + \bigvee_{i\in I}^{|I|-1} \tilde{x}_i.$$

Therefore

$$\sum_{i\in I}(a_i + c) = \sum_{i\in I} a_i + c + (|I| - 1)c = \bigvee_{i\in I}^{|I|-1} \tilde{x}_i + (|I| - 1)c + d$$
$$= \bigvee_{i\in I}\left(b_i + \sum_{k\in I\setminus\{i\}} (a_k + c)\right) + d.$$

Now from Lemma 2.2 it follows that $d = \bigwedge_{i\in I}(a_i + c \dot{-} b_i)$ exactly shadows $([a_i + c, b_i])_{i\in I}$. □

*Remark 2.4* If $\bigwedge_{i\in I}(a_i \dot{-} b_i)$ exactly shadows $(\tilde{x}_i)_{i\in I}$, then $\sum_{i\in I} a_i = \bigvee_{i\in I}^{|I|-1} \tilde{x}_i + \bigwedge_{i\in I} (a_i \dot{-} b_i)$. Hence $\inf_{i\in I} \tilde{x}_{i\in I} = [\bigwedge_{i\in I}(a_i \dot{-} b_i), 0]$.

## 3 Minimality criteria for exhausters

Differently from the above mentioned work of V.F. Demyanov, A.M. Rubinov and V. Roshchina (see [5–8,12]) we consider only exhausters with a finite index set for elements in a semigroup. To distinguish in this section elements of semigroup from functions, we denote by $\tilde{h}$ an element of a semigroup and by $h$ a function. Now we say that a set $I^*(\tilde{h}) = \{a_i\}_{i\in I}$ of elements of a semigroup $S$ is an *upper exhauster* of an element $\tilde{h} \in \widetilde{S}$, if $\tilde{h}$ can be represented in the form $\tilde{h} = \inf_{i\in I}[a_i, 0]$. Since $I$ is finite it follows from Theorem 1.7 that $\tilde{h}$ always exists and that:

$$\tilde{h} = \left[\sum_{i\in I} a_i, \bigvee_{i\in I} \sum_{k\in I\setminus\{i\}} a_k\right].$$

We say that exhauster $I^*(\tilde{h})$ is *minimal by inclusion*, if there exists no other upper exhauster $J^*(\tilde{h})$ of $\tilde{h}$ such that $J^*(\tilde{h}) \subset I^*(\tilde{h})$.

From Theorem 1.8 follows the criterion of minimality by inclusion.

**Theorem 3.1** *If $S$ is a commutative $\dot{q}$-semigroup, then exhauster $I^*(\tilde{h})$ is minimal by inclusion if and only if*

$$\bigvee_{i\in J} \sum_{k\in I\{i\}} a_i < \bigvee_{i\in I} \sum_{k\in I\{i\}} a_i,$$

*for any proper subset $J \subset I$.*

We say that exhauster $I^*(\tilde{h})$is $I$ *-minimal*, if for any $a_i' \leq a_i$, $i \in I$, $\tilde{h} = \inf_{i\in I}[a_i', 0]$ implies $a' = a_i$, $i \in I$.

From Lemma 2.2 and definition of $I$-minimality of exhausters we have

**Proposition 3.2** *If $S$ is $\dot{q}$-semigroup and $\bigwedge_{i\in I} a_i$ exactly shadows $([a_i, 0])_{i\in I}$, then exhauster $I^*(\tilde{h})$ is $I$-minimal if and only if $\bigwedge_{i\in I} a_i' = \bigwedge_{i\in I} a_i$ for any $a_i' \leq a_i$, $i \in I$, implies $a_i' = a_i$, $i \in I$.*

We observe that if for $a_i' \leq a_i$, $i \in I$, $\bigwedge_{i\in I} a_i' = \bigwedge_{i\in I} a_i$, then for every $j \in I$ we have $a_j' \wedge \bigwedge_{i\in I\setminus\{j\}} a_i = \bigwedge_{i\in I} a_i$.

Now from Theorem 2.1 and Theorem 2.3 about shadows and Proposition 3.2 we obtain a general criterion of $I$-minimality of exhausters.

**Theorem 3.3** *If $S$ is $\dot{q}$-semigroup and $(c, d) \in [\sum_{i\in I} a_i, \bigvee_{i\in I} \sum_{k\in I\setminus\{i\}} a_k]$, then exhauster $I^*(\tilde{h})$ is $I$-minimal if and only if*
*$\bigwedge_{i\in I}(a_i' + c) = \bigwedge_{i\in I}(a_i + c)$ for $a_i' \leq a_i$, $i \in I$, implies $a_i' = a_i$, $i \in I$.*

*Proof* Let $a_i' \leq a_i$, $i \in I$ and

$$\left[\sum_{i\in I} a_i', \bigvee_{i\in I} \sum_{k\in I\setminus\{i\}} a_i'\right] = \left[\sum_{i\in I} a_i, \bigvee_{i\in I} \sum_{k\in I\setminus\{i\}} a_k\right].$$

By Theorem 2.3 we have

$$\bigwedge_{i\in I}(a_i' + c) = \bigwedge_{i\in I}(a_i + c) = d$$

shadows $([a_i' + c, 0])_{i\in I}$ and $([a_i + c, 0])_{i\in I}$. Moreover

$$\inf_{i\in I}[a_i' + c, 0] = \inf_{i\in I}[a_i + c, 0] = \left[\sum_{i\in I} a_i, \bigvee_{i\in I} \sum_{k\in I\setminus\{i\}} a_k\right]. \qquad \square$$

We say that the sequence $(\tilde{x}_j)_{j\in J}$ is *finer* then the sequence $(\tilde{x}_i)_{i\in I}$ if for every $j \in J$ there exists $i \in I$ such that $\tilde{x}_j \le \tilde{x}_i$.

We say that upper exhauster $I^*(\tilde{h})$ is *minimal*, if there exists no other upper exhauster $J^*(\tilde{h})$ of $\tilde{h}$ finer then $I^*(\tilde{h})$.

**Theorem 3.4** *If $S$ is $\dot{q}$-semigroup, then upper exhauster $I^*(\tilde{h})$ is minimal if and only if is minimal by inclusion and is $I$-minimal. It is equivalent that exhauster $I^*(\tilde{h})$ satisfies the following condition:*
$\bigvee_{i\in J}\sum_{k\in I\setminus\{i\}} a_k < \bigvee_{i\in I}\sum_{k\in I\setminus\{i\}} a_k$ *for every proper subset* $J \subset I$.
*If* $\bigwedge_{i\in I}(a'_i + c) = \bigwedge_{i\in I}(a_i + c)$ *for* $a'_i \le a_i$, $i \in I$ *and*
$(c, d) \in [\sum_{i\in I} a_i, \bigvee_{i\in I}\sum_{k\in I\setminus\{i\}} a_k]$, *then* $a'_i = a_i$ *for* $i \in I$.

*Proof* Let $\tilde{h} = \inf_{i\in I}[a_i, 0] = \inf_{j\in J}[a'_j, 0]$, where $J^*(\tilde{h})$ is finer then $I^*(\tilde{h})$. Now given

$$(c, d) \in \left[\sum_{i\in I} a_i, \bigvee_{i\in I}\sum_{k\in I\setminus\{i\}} a_k\right],$$

then

$$d = \bigwedge_{i\in I}(a_i + c) = \bigwedge_{j\in J}(a'_j + c).$$

Let $J_k = \{j \in J \mid a'_j \le a_k\}$, $k \in I$. By assumption $J_k \ne \emptyset$ for some $k \in I$. Denote $I_0 = \{k \in I \mid J_k \ne \emptyset\}$. Now given arbitrary $a'_j$, there exists $k \in I$ such that $a'_j \le a_k$. Therefore $d = \bigwedge_{j\in J}(a'_j + c) \le a'_j + c$, hence $d \le (a'_j + c) \wedge \bigwedge_{i\in I\setminus\{j\}}(a_i + c)$. Now we consider

$$a''_i = \begin{cases} a'_j & \text{for } i = k, \\ a_i & \text{for } i \in I \setminus \{k\}. \end{cases}$$

Then $d = \bigwedge_{i\in I}(a''_i + c)$ and $a''_i \le a_i$ for $i \in I$. Since exhauster $I^*(\tilde{h})$ is $I$-minimal, $a'_j = a_k$ and from arbitrary $a'_j$ we have $\{a'_j\}_{j\in J} = \{a_i\}_{i\in I_0}$. But exhauster $I^*(\tilde{h})$ is minimal by the inclusion, hence $I_0 = I$. □

## 4 Reduction

For an ordered commutative semigroup $(S, +, \le)$ we call a pair $(a, b) \in S^2$ is *convex* if $a + b \le a \vee b + a \wedge b$.

**Lemma 4.1** *Let $a, b, f, p \in S$, pair $(a, p)$ be convex and $b = a \vee p$, $f = a \wedge p$. Then the pairs $(a, b)$ and $(f, p)$ are equivalent.*

*Proof* From the convexity of pair $(a, p)$ we have $a + p = a \vee p + a \wedge p$. Therefore $a + p = b + f$ and we obtain $(a, b) \sim (f, p)$. □

**Theorem 4.2** *Let $m_1 \vee a_2 = f \vee a_2$, $(m_1, p_1)$ be a convex pair and $a_1 = m_1 \vee p_1$, $f = m_1 \wedge p_1$. Moreover let $(p_1, m_1 \vee a_2)$ be a convex pair. Then pairs $(a_1 + a_2, a_1 \vee a_2)$ and $(m_1 + a_2, m_1 \vee a_2)$ are equivalent,* $\inf\{[a_1, 0], [a_2, 0]\} = \inf\{[m_1, 0], [a_2, 0]\}$.

*Proof* Let $a = m_1 \vee a_2, b = a_1 \vee a_2$. Then from the Lemma 4.1 it follows that $(a, b)$ is equivalent $(f, p_1)$. Therefore $m_1 \vee a_2 + p_1 = a_1 \vee a_2 + f$ but $a_1 + f = p_1 + m_1$ hence

$$(f, p_1) \sim (m_1, a_1) \sim (m_1 + a_2, a_1 + a_2) \sim (m_1 \vee a_2, a_1 \vee a_2). \qquad \square$$

**Theorem 4.3** *Let $f \vee g = m_1 \vee m_2$, pairs $(m_1, p_1), (m_2, p_2)$, are convex and $a_1 = m_1 \vee p_1, f = m_1 \wedge p_1, g = m_2 \wedge p_2$. Moreover let pairs $(p_1, m_1 \vee a_2)$ and $(p_2, m_2 \vee a_1)$ are convex. Then pairs $(a_1 + a_2, a_1 \vee a_2)$ and $(m_1 + m_2, m_1 \vee m_2)$ are equivalent.*

*Proof* By Theorem 4.2 it follows that

$$(a_1 + a_2, a_1 \vee a_2) \sim (m_1 + a_2, m_1 \vee a_2)$$

and using again Theorem 3.4 we get

$$(a_1 + a_2, a_1 \vee a_2) \sim (m_1 + m_2, m_1 \vee m_2). \qquad \square$$

## 5 Applications to the Minkowski–Rådström–Hörmander lattice

Let $X$ be a Hausdorff topological vector space over the field $\mathbb{R}$ and $\mathcal{B}(X)$ be the family of all nonempty closed bounded convex subsets of $X$. For $A, B, C, D \in \mathcal{B}(X)$ we have $A + B = \{a + b \mid a \in A, b \in B\}$, $A \dot{+} B = \overline{A + B}$, $A \vee B = \overline{\text{conv}}\,(A \cup B)$, $A \dot{-} B = \{x \in X \mid x + B \subset A\}$, $(A, B) \sim (C, D)$ if and only if $A \dot{+} D = B \dot{+} C$, $[A, B] = [(A, B)]_{\sim}$, $[A, B] \le [C, D]$ if and only if $A \dot{+} D \subset B \dot{+} C$, $[A, B] + [C, D] = [A \dot{+} C, B \dot{+} D]$, $[A, B] \vee [C, D] = \sup\{[A, B], [C, D]\} = [(A \dot{+} D) \vee (B \dot{+} C), B \dot{+} D]$. The quotient space $\widetilde{X} = \mathcal{B}^2(X)/_{\sim}$ is called the *Minkowski–Rådström–Hörmander* (M–R–H) *lattice* over $X$. Moreover, $(\widetilde{X}, +, \le)$ is a topological vector lattice. For the definition of M–R–H lattice we refer to [31], and to the paragraph 3.4 in [21].

There exists an interesting duality with the space of differences of sublinear functions. Since we need this duality only in the finite dimensional case, we will consider instead of $\mathcal{B}(X)$ the semi-subgroup $\mathcal{K}(X)$ of all compact convex subsets of $X$. Endowed with the order $[A, B] \preceq [C, D] \iff A + D \subseteq B + C$ the space $\left(\mathcal{K}^2(X')/_{\sim}, \preceq\right)$ is a sublattice of $\widetilde{X}$.

Now we consider the convex cone of all weak-*-continuous sublinear functions $\mathcal{P}(X')$ on the dual $X'$ of $X$ and put $\mathcal{D}(X') = \{\varphi = p - q \mid p, q \in \mathcal{P}(X)\}$. With respect to the pointwise ordering of functions given by $\varphi \le \psi$ if and only if $\varphi(x) \le \psi(x)$ holds for every $x \in X'$, the space $(\mathcal{D}(X'), \le)$ is a vector lattice. Since the sublinear function $p : X \longrightarrow \mathbb{R}$ is weak-*-continuous its subdifferential $\partial p\big|_0 \subset X$ is a compact convex subset of $X$. Now the assignment:

$$\mathcal{D}(X) \longrightarrow \mathcal{K}^2(X')/_{\sim} \quad \text{with} \quad \varphi \mapsto [\varphi] = \{(\partial p|_0, \partial q|_0) \mid \text{ with } \varphi = p - q,\ \ p, q \in \mathcal{P}(X)\}$$

is a lattice isomorphism, called *Minkowski duality* (see [21]).

The Minkowski–Rådström–Hörmander lattice [19,25] is very useful in studying bounded-valued correspondences [4,14], in quasidifferential calculus [12,13,16,17,21,23,30,32] and in calculating Aumann–Integral [2]. The M–R–H lattices were studied also in a number of papers, for example [15,24,31].

Since semigroup $S = (\mathcal{B}(X), \dot{+}, \subset)$ is a $\dot{q}$-semigroup then all results concerning upper exhausters which we obtained in the previous sections can be applied to the semigroup $S = \mathcal{B}(X)$.

First we present a result concerning shadowing. It is easy to observe the following fact:

**Lemma 5.1** *Let $X$ be a vector space, $(a_i)_{i\in I}, (b_i)_{i\in I} \subset X$ and $(\alpha_i)_{i\in I} \subset \mathbb{R}$. Then*

$$\left(\sum_{i\in I}\alpha_i\right)\left(\sum_{i\in I}a_i\right) = \sum_{i\in I}\alpha_i\left(b_i + \sum_{k\in I\setminus\{i\}} a_k\right) + \sum_{i\in I}\alpha_i(a_i - b_i).$$

**Theorem 5.2** *Let $X$ be a topological vector space. Then the set $C \in \mathcal{B}(X)$ shadows $([A_i, B_i])_{i\in I} \subset \mathcal{B}(X)\times\mathcal{K}(X)\big/_{\sim} \subset \tilde{X} = \mathcal{B}^2(X)\big/_{\sim}$ if and only if for every $a_i \in A_i$, $i \in I$ there exists $b_i \in B_i$, $i \in I$ such that*

$$C \cap \bigvee_{i\in I}(a_i - b_i) \neq \emptyset. \qquad (*)$$

*Proof* Suppose that $\sum_{i\in I} a_i \subset \bigvee_{i\in I}^{|I|-1} \tilde{x}_i + C$. Since for every $a_i \in A_i$, $i \in I$ we have

$$B_i + \sum_{i\in I} A_i \subset \bigvee_{i\in I}^{|I|-1} \tilde{x}_i \dotplus C + (B_i - a_i),$$

it follows that

$$\bigvee_{i\in I}\left(B_i + \sum_{k\in I\setminus\{i\}} A_k\right) \subset \bigvee_{i\in I}\left(B_i + \sum_{k\in I\setminus\{i\}} A_k\right) \dotplus C + \bigvee_{i\in I}(B_i - a_i).$$

Now from the order cancellation law we have $0 \in C + \bigvee_{i\in I}(B_i - a_i)$, and hence there exist $c \in C$ and $b_i \in B_i$, $i \in I$ such that $c = \sum_{i\in I}\alpha_i(a_i - b_i)$ for a some convex combination $(\alpha_i) \subset \mathbb{R}$ with $\sum_{i\in I}\alpha_i = 1$.

Now let $(\tilde{x}_i)_{i\in I}$ satisfy condition $(*)$. Then it follows from Lemma 5.1 it that $\sum_{i\in I} a_i \subset \bigvee_{i\in I}^{|I|-1} \tilde{x}_i + C$. □

**Corollary 5.3** *Let $X$ be a topological vector space. Then the set $C \in \mathcal{B}(X)$ shadows $(A_i)_{i\in I} \subset \mathcal{B}(X)$ if and only if for every $a_i \in A_i$, $i \in I$ one has $C \cap \bigvee_{i\in I} a_i \neq \emptyset$.*

## 6 Minimality of exhausters in $\mathcal{B}(X)$

From the above sections we deduce for the case of the semigroup $S = (\mathcal{B}(X), \dotplus, \subset)$ the following results: For example Theorem 3.4 can be stated as:

**Theorem 6.1** *If $S = \mathcal{B}(X)$, then upper exhauster $I^*(h)$ is minimal if and only if it is minimal by inclusion and is $I$-minimal or equivalently that the exhauster $I^*(h)$ satisfies the following condition:*

*$\bigvee_{i\in J}\sum_{k\in I\setminus\{i\}} A_k \subset \bigvee_{i\in I}\sum_{k\in I\setminus\{i\}} A_k$ for every proper subset $J \subset I$. If $\bigwedge_{i\in I}(A_i'+C) = \bigwedge_{i\in I}(A_i + C)$ for $A_i' \subset A_i$, $i \in I$ and $(C, D) \in [\sum_{i\in I} A_i, \bigvee_{i\in I}\sum_{k\in I\setminus\{i\}} A_k]$, then $A_i' = A_i$ for $i \in I$.*

Now we observe that for upper exhauster $I^*(\tilde{h})$ we have $\tilde{h} = [A, B] \in \tilde{X}$, where $(A, B) \in [\sum_{i\in I} A_i, \bigvee_{i\in I}\sum_{k\in I\setminus\{i\}} A_i]$. Denote $A = \underline{\partial} h$, $B = \bar{\partial} h$. Using the Minkowski duality we obtain that (see [5–7,29]) $I^*(\tilde{h}) = E^*(h) = \{A_i \mid i \in I\}$, where $h = \inf_{i\in I} p_{A_i} =$

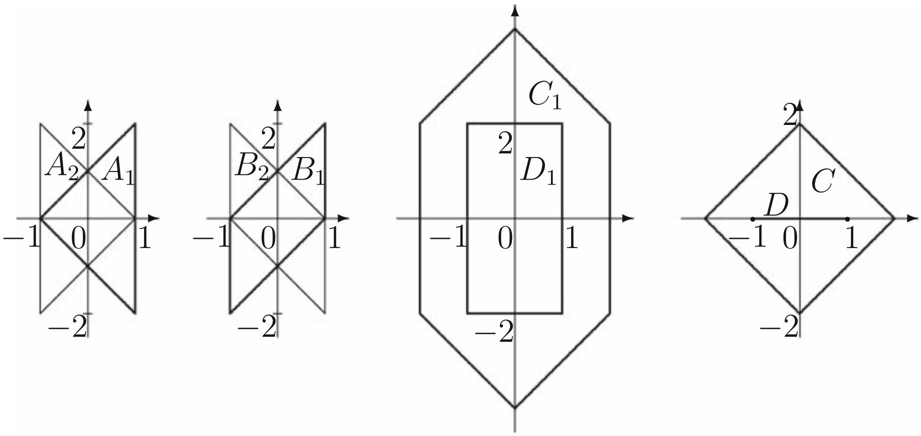
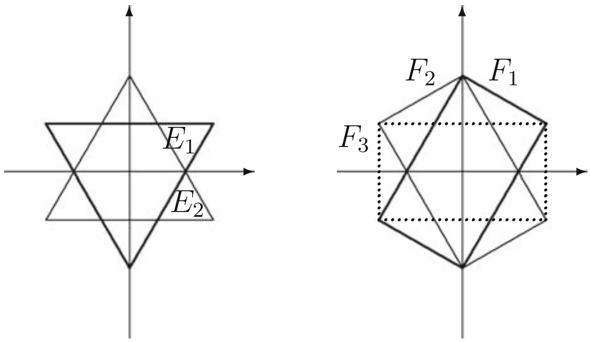



**Fig. 2** Two different minimal upper exhausters (Case 1)

**Fig. 3** Two different minimal upper exhausters (Case 2)




$p_A - p_B \in \mathcal{D}(X)$. Of course minimality of upper exhauster $E^*(h)$ is equivalent to minimality of upper exhauster $I^*(\tilde{h})$.

Now we give an example of two different minimal upper exhausters of the same function. This answers a question posed by Vera Roshchina [29].

*Example 6.2* Consider two triangles $A_1 = (1, 2) \vee (1, -2) \vee (-1, 0)$ and $A_2 = (-1, 2) \vee (-1, -2) \vee (1, 0)$ in $\mathcal{B}(\mathbb{R}^2)$. Let $h = \min(p_{A_1}, p_{A_2})$. Then $\{A_1, A_2\} = I^*(\tilde{h})$ is an upper exhauster of the function $h$. Notice that $h = p_{C_1} - p_{D_1} = p_C - p_D$, where $C_1 = A_1 + A_2$ and $D_1 = A_1 \vee A_2$ and $(C, D)$ is a minimal pair of convex sets equivalent to $(C_1, D_1)$. Since the function $h$ is not convex, any upper exhauster of $h$ has to contain at least two sets. Then $I^*(\tilde{h})$ is inclusion-minimal. Assume that $K^*(\tilde{h}) = \{E_\iota\}_{\iota \in I}$ is an exhauster finer than $I^*(\tilde{h})$ and that $E_\lambda \subsetneq A_1$ for some $\lambda \in I$. Then the set $E_\lambda$ does not contain one of the vertices of the triangle $A_1$. Hence $h(x) = \inf_{\iota \in I} p_{E_\iota}(x) \le p_{E_\lambda}(x) < p_{A_1}(x) = \min(p_{A_1}(x), p_{A_2}(x)) = h(x)$ for $x = (1, 0), (0, 1)$ or $(0, -1)$ which is impossible and contradicts the assumption that the upper exhauster $K^*(\tilde{h})$ is essentially finer than $I^*(\tilde{h})$. Therefore, the upper exhauster $I^*(\tilde{h})$ is minimal. On the other hand, the upper exhauster $J^*(\tilde{h}) = \{B_1, B_2\}$ is also minimal. Then the uniqueness of minimal upper exhausters does not hold (Fig. 2).

Another example of two different minimal upper exhauster is provided in Fig. 3.

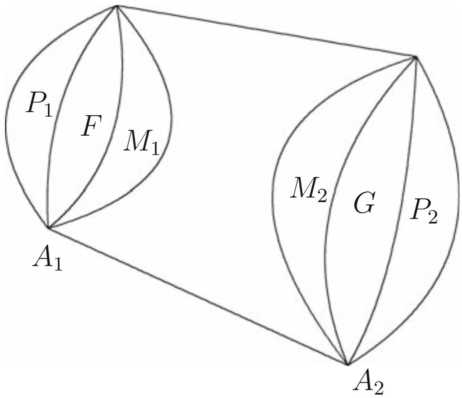



**Fig. 4** Illustration of the reduction technique

## 7 Special reduction techniques

For the semigroup $S = (\mathcal{B}(X), \dot{+}, \subset)$ we obtain from the considerations in Sect. 5 the following reduction techniques, which are summarized now:

**Theorem 7.1** *Let* $M_1 \vee A_2 = F \vee A_2$, $A_1 = M_1 \cup P_1$, $F = M_1 \cap P_1$. *Moreover let pair* $(P_1, M_1 \vee A_2)$ *be convex. Then pairs* $(A_1 + A_2, A_1 \vee A_2)$ *and* $(M_1 + A_2, M_1 \vee A_2)$ *are equivalent and* $\inf\{[A_1, 0], [A_2, 0]\} = \inf\{[M_1, 0], [A_2, 0]\}$ *(Fig. 2).*

**Theorem 7.2** *Let* $F \vee G = M_1 \vee M_2$, $A_1 = M_1 \cup P_1$, $A_2 = M_2 \cup P_2$, $F = M_1 \cap P_1$, $G = M_2 \cap P_2$. *Moreover let pairs* $(P_1, M_1 \vee A_2)$, $(P_2, M_2 \vee A_1)$ *be convex. Then pairs* $(A_1 + A_2, A_1 \vee A_2)$ *and* $(M_1 + M_2, M_1 \vee M_2)$ *are equivalent (Fig. 4).*

**Acknowledgements** The authors would like to thank Vladimir F. Demyanov for stimulating discussions on the relation between exhausters and quasi-differentials.

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